MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN3A. Matrix Transformations.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights parts of the text that are used in the lecture sessions.

A common problem in matrix algebra is finding some subset W of a given set of vectors V such that W is linearly independent and $\text{span}(V) = \text{span}(W)$, i.e. every linear combination of vectors in V can be expressed as a unique linear combination of vectors in W.

Theorem 3A.1. Linearly Independent Spanning Sets

Let $V = \{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$ be a set of vectors in $\mathbb{R}^n$. Then, there exists some subset $W \subseteq V$ such that W is linearly independent and $\operatorname{span} V = \operatorname{span} W$. Note that this subset W is generally not unique. There are two methods we can use to find W:

(a) Remove vectors in V one-by-one until the resultant set $W = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is linearly independent, with $k+1$ the index of the last removed vector.

(b) Let $\mathbf{M}$ be a row-echelon form of the matrix $\mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_m} \\ | & | & & | \end{pmatrix}$.

Then, W can be generated using

$$W = \left\{\mathbf{v_i} \in V : \mathsf{col}_i\, \mathbf{M} \text{ contains a pivot}\right\}$$

where each column of $\mathbf{M}$ corresponds to a column in $\mathbf{V}$ and therefore, a vector in V.

Note that when V is linearly independent, $W = V$.

Definition 3A.2. Linear Transformations

A **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function/transformation such that for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars $k \in \mathbb{R}$,

$$T(k\mathbf{v} + \mathbf{w}) = kT(\mathbf{v}) + T(\mathbf{w})$$

When $m = n$, we say that T is a **linear operator** or simply a **transformation** on $\mathbb{R}^n$.

Example 3A.2.1. Differentiation and Integration are linear functions on certain function spaces.

Here's a quick criteria to check if a transformation is not linear. Note that this result is not an equivalence.

Theorem 3A.3. Necessary Conditions on 0

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation. If $T(\mathbf{0}) \neq \mathbf{0}$, then T is not a linear transformation.

Example 3A.3.1. The function $f(x) = mx + b$ with $b \neq 0$ is **not** a linear transformation. More generally, translations are not linear.

Theorem 3A.4. Linear Transforms as Matrix Multiplication

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there exists a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$ and $T(\mathbf{x}) \in \mathbb{R}^m$ are both represented as column vectors. The matrix $\mathbf{A}$ is unique and is given by

$$\mathbf{A} = \begin{pmatrix} | & | & & | \\ T(\mathbf{e_1}) & T(\mathbf{e_2}) & \cdots & T(\mathbf{e_n}) \\ | & | & & | \end{pmatrix} \qquad \text{with} \qquad T(\mathbf{e_i}) \in \mathbb{R}^m$$

When $\mathbf{A}$ exists, we say that $\mathbf{A}$ is the left multiplication matrix of T or T is given by left multiplication by $\mathbf{A}$.

Some function operations have matrix analogs. We typically focus on three operations.

Theorem 3A.5. Identity Transformations

Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation mapping $\mathbf{x}$ to itself. Then, the left multiplication matrix of I is given by $\mathbf{A} = \mathbf{I_n}$.

Theorem 3A.6. Function Composition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations with left multiplication matrices $\mathbf{A} \in \mathbb{R}^{n\times m}, \mathbf{B} \in \mathbb{R}^{m\times k}$ respectively. Then, the composition by applying T and then S is given by $(S \circ T)(\mathbf{x}) = \mathbf{B}\mathbf{A}\mathbf{x}$.

Theorem 3A.7. Invertibility by Matrices

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if its left multiplication matrix $\mathbf{A}$ is invertible. Then, the inverse transformation is given by $T^{-1}(\mathbf{y}) = \mathbf{A}^{-1}\mathbf{y}$

The result below is one of the most powerful tools for finding the left multiplication matrix $\mathbf{A}$. In linear algebra, this result is often known as a change of basis and is stated a bit differently.

Theorem 3A.8. Change of Basis

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with left multiplication matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$. Let $V = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be a basis for $\mathbb{R}^n$ such that the values $T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})$ are known. Then, $\mathbf{A}$ can be identified by:

$$\mathbf{A} \begin{pmatrix} | & | & & | \\ \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ T(\mathbf{v_1}) & T(\mathbf{v_2}) & \cdots & T(\mathbf{v_n}) \\ | & | & & | \end{pmatrix}$$

Equivalently,

$$\mathbf{A} = \begin{pmatrix} | & | & & | \\ T(\mathbf{v_1}) & T(\mathbf{v_2}) & \cdots & T(\mathbf{v_n}) \\ | & | & & | \end{pmatrix} \begin{pmatrix} | & | & & | \\ \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \\ | & | & & | \end{pmatrix}^{-1}$$

We will consider several linear transformations on $\mathbb{R}^2$ and $\mathbb{R}^3$ and express those transformations using left multiplication matrices.

Theorem 3A.9. Scaling Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a transformation that scales each vector $\mathbf{x}$ by a given factor of $k \in \mathbb{R}$, i.e. $\mathbf{x} \mapsto k\mathbf{x}$. Then, $\mathbf{A}$ is given by $\mathbf{A} = k\mathbf{I_n}$.

Theorem 3A.10. Rotations on $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector $\mathbf{x}$ counterclockwise by θ radians. Then, the left multiplication matrix $\mathbf{R}_\theta$ of T is given by $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. We call $\mathbf{R}_\theta$ a **rotation matrix**.

We can also consider rotations on $\mathbb{R}^3$. However, we will only consider rotations along the x, y, and z-axes since finding rotation matrices by some arbitrary axis is a non-trivial problem.

Theorem 3A.11. Rotations on $\mathbb{R}^3$

Let $\mathbf{R_x}, \mathbf{R_y}, \mathbf{R_z} \in \mathbb{R}^3$ be rotation matrices about the x, y, and z-axes by θ radians respectively. Then,

$$\mathbf{R_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad \mathbf{R_z} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \qquad \mathbf{R_z} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with θ the angle counterclockwise the given axis by the right-hand rule.

We consider transformations on $\mathbb{R}^2$ relative to lines.

Theorem 3A.12. Basis of $\mathbb{R}^2$ relative to a Line L

Let L be a line spanned by $\mathbf{d} = \begin{pmatrix} a \\ b \end{pmatrix}$, i.e. L admits the parametrization $L(t) = t\mathbf{d}$.

Then, $\mathbf{b} = \begin{pmatrix} -b \\ a \end{pmatrix}$ is orthogonal to $\mathbf{d}$ and $\{\mathbf{d}, \mathbf{b}\}$ is a basis for $\mathbb{R}^2$.

Theorem 3A.13. Reflection across Lines on $\mathbb{R}^2$

Let L be a line spanned by $\mathbf{d} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then, the left multiplication matrix $\mathbf{A}$ of the transformation reflecting the point $\mathbf{x}$ across the line L is determined by

$$\mathbf{A}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \text{or equivalently} \qquad \mathbf{A} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1}$$

Theorem 3A.14. Orthogonal Projection along a Line on $\mathbb{R}^2$

Let L be a line spanned by $\mathbf{d} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then, the left multiplication matrix $\mathbf{A}$ of the transformation projecting the arrow $\mathbf{x}$ onto the line L is determined by

$$\mathbf{A}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \qquad \text{or equivalently} \qquad \mathbf{A} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1}$$

Then, transformations in $\mathbb{R}^3$ related to planes.

Theorem 3A.15. Basis of $\mathbb{R}^3$ relative to a plane P

Let P be a plane with normal vector $\mathbf{N}$. Let $\mathbf{d_1}, \mathbf{d_2}$ be non-parallel direction vectors of P. Then, the set $\{\mathbf{N}, \mathbf{d_1}, \mathbf{d_2}\}$ is a basis for $\mathbb{R}^3$.

Theorem 3A.16. Reflection across a Plane

Let P be a plane with normal vector $\mathbf{N}$ and non-parallel direction vectors $\mathbf{d_1}, \mathbf{d_2}$. Then, the left multiplication matrix $\mathbf{A}$ of the transformation reflecting a point $\mathbf{x}$ across the plane P is determined by

$$\mathbf{A}\begin{pmatrix} | & | & | \\ \mathbf{N} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ -\mathbf{N} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix} \qquad \text{or equivalently} \qquad \mathbf{A} = \begin{pmatrix} | & | & | \\ -\mathbf{N} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix}\begin{pmatrix} | & | & | \\ \mathbf{N} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix}^{-1}$$

Theorem 3A.17. Projection along a Plane

Let P be a plane with normal vector $\mathbf{N}$ and non-parallel direction vectors $\mathbf{d_1}, \mathbf{d_2}$. Then, the left multiplication matrix $\mathbf{A}$ of the transformation projecting an arrow $\mathbf{x}$ on to the plane P is determined by

$$\mathbf{A}\begin{pmatrix} | & | & | \\ \mathbf{N} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \mathbf{0} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix} \qquad \text{or equivalently} \qquad \mathbf{A} = \begin{pmatrix} | & | & | \\ \mathbf{0} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix}\begin{pmatrix} | & | & | \\ \mathbf{N} & \mathbf{d_1} & \mathbf{d_2} \\ | & | & | \end{pmatrix}^{-1}$$